A short note on differentiation using the chain rule repeatedly

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Example 1

Given $y = \ln(\ln x)$, find $\frac{dy}{dx}$.

Method 1

Let $u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$

$y = \ln u$, $\frac{dy}{du} = \frac{1}{u}$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u} \times \frac{1}{x} = \frac{1}{\ln x} \times \frac{1}{x} = \frac{1}{x \ln x}$$

Method 2

$$\frac{d[\ln(\ln x)]}{dx} = \frac{d[\ln(\ln x)]}{d(\ln x)} \times \frac{d(\ln x)}{dx} = \frac{1}{\ln x} \times \frac{1}{x} = \frac{1}{x \ln x}$$

Example 2

Given $y = \ln[\sec(2x - 3)]$, find $\frac{dy}{dx}$.

We first employ the method 2 of example 1.

$$\frac{d\{\ln[\sec(2x-3)]\}}{dx} = \frac{d\{\ln[\sec(2x-3)]\}}{d[\sec(2x-3)]} \times \frac{d[\sec(2x-3)]}{d(2x-3)} \times \frac{d(2x-3)}{dx}$$

$$= \frac{1}{\sec(2x-3)} \times [\sec(2x - 3)\tan(2x - 3)] \times 2 = \mathbf{2\tan(2x - 3)}$$

We devise the following rule for function with many **layers**.

(1)	Differentiate the outermost layer.
(2)	Write down in (1) those inside the outermost layer as if it is the "x" in differentiation rule.
(3)	Cancel the outermost layer.
(4)	The result in (2) multiplies by the derivative of (3).
(5)	Stop if the differentiation ends, otherwise go to (1).

For this example the outermost layer is ln( ), the derivative is of the form $\frac{1}{(\ )}$.

The inside is $\sec(2x - 3)$.

So we write:

$$\frac{d\{\ln[\sec(2x-3)]\}}{dx} = \frac{1}{\sec(2x-3)}\frac{d}{dx}[\sec(2x-3)]$$

Do the same thing for $\frac{d}{dx}[\sec(2x-3)]$.

$$\frac{d\{\ln[\sec(2x-3)]\}}{dx} = \frac{1}{\sec(2x-3)}[\sec(2x-3)\tan(2x-3)]\frac{d}{dx}(2x-3)$$

$$= \frac{1}{\sec(2x-3)} \times [\sec(2x-3)\tan(2x-3)] \times 2$$

$$= \mathbf{2\tan(2x-3)}$$

Can you follow the rule for the example below?

Example 3

Given $y = \tan\left[\ln\sqrt{\sin 3x}\right]$, find $\frac{dy}{dx}$.

$$\frac{d\{\tan\left[\ln\sqrt{\sin 3x}\right]\}}{dx} = \sec^2\left[\ln\sqrt{\sin 3x}\right]\frac{d}{dx}\left[\ln\sqrt{\sin 3x}\right]$$

$$= \sec^2\left[\ln\sqrt{\sin 3x}\right] \times \frac{1}{\sqrt{\sin 3x}} \times \frac{d}{dx}\sqrt{\sin 3x}$$

$$= \sec^2\left[\ln\sqrt{\sin 3x}\right] \times \frac{1}{\sqrt{\sin 3x}} \times \frac{1}{2\sqrt{\sin 3x}} \times \frac{d}{dx}(\sin 3x)$$

$$= \sec^2\left[\ln\sqrt{\sin 3x}\right] \times \frac{1}{\sqrt{\sin 3x}} \times \frac{1}{2\sqrt{\sin 3x}} \times \cos 3x \times \frac{d}{dx}(3x)$$

$$= \sec^2\left[\ln\sqrt{\sin 3x}\right] \times \frac{1}{\sqrt{\sin 3x}} \times \frac{1}{2\sqrt{\sin 3x}} \times \cos 3x \times 3$$

$$= \boldsymbol{\frac{3}{2}\cot 3x \sec^2\left[\ln\sqrt{\sin 3x}\right]}$$